B.Sc Physics Part-II,

Paper-IV, Group-A

## Laplace equation in spherical coordinates I:

Laplace equation is given by

$$\nabla^2 V = 0$$

In spherical coordinates $(r, \theta, \phi)$, Laplace equation is written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad \text{--- (1)}$$

We use separation of variables to find the solution of above equation. We assume that potential is written in the product form as

$$V(r,\theta,\phi) = \frac{R(r)}{r} P(\theta)\, Q(\phi) \quad \text{--- (2)}$$

To simplify the algebra we have included an extra factor '$\frac{1}{r}$'. Next, using eq. (2) in equation (1)

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left[r \cdot \frac{R(r)}{r} P(\theta) Q(\phi)\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial}{\partial \theta}\left(\frac{R(r)}{r} P(\theta) Q(\phi)\right)\right]$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}\left[\frac{R(r)}{r} P(\theta) Q(\phi)\right] = 0$$

or $$P(\theta) Q(\phi) \frac{1}{r}\frac{\partial^2 R(r)}{\partial r^2} + \frac{R(r)}{r} Q(\phi)\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P(\theta)}{\partial \theta}\right)$$

$$+ \frac{R(r)}{r} P(\theta) \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 Q(\phi)}{\partial \phi^2} = 0$$

Next, we multiply above expression by $\dfrac{r^3 \sin\theta}{R(r)P(\theta)Q(\phi)}$

we obtain,

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right) +$$

$$+ \frac{1}{Q(\phi)}\frac{d^2Q(\phi)}{d\phi^2} = 0 \quad \text{--- (3)}$$

The third term is independent of r and $\theta$, thus it ~~also~~ must be equal to a constant.

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right) - m^2 = 0 \quad \text{--- (4)}$$

and $$\frac{1}{Q(\phi)}\frac{d^2Q(\phi)}{d\phi^2} = -m^2 \quad \text{--- (5)}$$

Eq. (5) can be solved.

$$\frac{d^2Q(\phi)}{d\phi^2} = -m^2Q(\phi)$$

$$\left(\frac{d^2}{d\phi^2} + m^2\right)Q(\phi) = 0$$

Solution is $$Q(\phi) = A_m e^{im\phi} + B_m e^{-im\phi}, \quad m \neq 0$$

for $m = 0$, soln $\rightarrow Q(\phi) = A_0 + B_0\phi$ --- (6)

Again taking Eq. (4)

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right) - m^2 = 0$$

dividing by $\sin^2\theta$,

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{\sin\theta\, P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=0 \quad \text{——} (7)$$

The last term and first term are independent and therefore we can write

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}=l(l+1)=0 \quad \text{——} (8)$$

where $\frac{1}{\sin\theta\, P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-l(l+1)$ —— (9)

Now eqs. (8) & (9) can be written as

$$\frac{d^2R(r)}{dr^2}=l(l+1)\frac{R(r)}{r^2}$$

and $\frac{d}{d\theta}\left[\sin\theta\frac{dP(\theta)}{d\theta}\right]+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\sin\theta\, P(\theta)=0$

For the last equation we take $x=\cos\theta$, which gives $\frac{d}{d\theta}=-\sqrt{1-x^2}\frac{d}{dx}$. Therefore, we write

$$\frac{d^2R(r)}{dr^2}=l(l+1)\frac{R(r)}{r^2} \quad \text{——} (10)$$

and $\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P(x)=0$ —— (11)

In the next lecture we will solve equation (10) and (11).